CLASSIFYING *-HOMOMORPHISMS

JOINT WORK WITH J. GABE, C. SCHAFHAUSER,
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INTRODUCTION

Theorem ("Many hands")

C^*-algebras that are unital, simple, separable, nuclear, regular, and satisfy the UCT, are classified by K-theory and traces.

- C^*-analog of the classification of injective von Neumann factors: Murray-von Neumann, Connes, Haagerup.
- No traces: Kirchberg-Phillips (1990s).
- We focus only on the case $T(A) \neq \varnothing$.
- Classifying invariant:

$$\mathrm{Ell}(A) := \Big(K_0(A),\, K_0(A)_+,\, [1_A]_0,\, K_1(A),\\ T(A),\, T(A) \times K_0(A) \to \mathbb{R}\Big)$$

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Setup: G, a countable discrete group, acting on X, a compact metric space.

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- ...nuclear, if G is amenable;
- ...in the UCT class, if G is amenable;
- ...regular, if X is finite dimensional and G is f.g. nilpotent.

THE ORIGINAL ROAD TO CLASSIFICATION ($\sim$2017)

Impossible to summarize decades of work in a slide.

Some recent components:

Classification of "model" algebras

- Gong-Lin-Niu '15: classified C^*-algebras with a certain internal tracial approximation structure.
- The class exhausts range of $\mathrm{Ell}(-)$.

Realizing the approximations

- Elliott-Gong-Lin-Niu '15: abstract conditions on a C^*-algebra $\Rightarrow$ concrete tracial approximations of GLN.
- Tikuisis-White-Winter '17: the abstract conditions are the ones stated in the classification theorem.

A DIFFERENT APPROACH

We develop an alternate route to classification: beginning with von Neumann algebraic techniques inspired by work of Connes and Haagerup, we extend the KK-theoretic techniques recently developed by Schafhauser to prove classification theorems in an abstract setting.

While tracial approximations are not used in this approach, the broad roadmap used in that setting will guide us.

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We will (mostly) ignore the difficulties that arise from non-separability or the lack of a unit.

CLASSIFYING MORPHISMS & ALGEBRAS

Rough scheme

Produce invariant $\mathrm{inv}(-)$ s.t.
(with abstract hypotheses on A, B):

- (existence)
  $\alpha\colon \mathrm{inv}(A) \to \mathrm{inv}(B) \implies \exists\, \varphi\colon A \to B$ s.t. $\mathrm{inv}(\varphi) = \alpha$;
- (uniqueness)
  $\varphi, \psi\colon A \to B$ and $\mathrm{inv}(\varphi) = \mathrm{inv}(\varphi) \implies \varphi \approx_u \psi$.

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Intertwining: $\mathrm{inv}(A) \cong \mathrm{inv}(B) \implies A \cong B$.

inv($-$) will be more complicated than Ell($-$).

Also want:

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B) \text{ yields } \mathrm{inv}(A) \cong \mathrm{inv}(B).$$

ONE INGREDIENT OF INV(—): “ALGEBRAIC” K_1

Thomsen-Nielsen (early 90s): different proof of Elliott’s classification of simple unital $A\mathbb{T}$ algebras. Need refined inv.

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$CU^\infty(A)$ is the *closure* of the commutator subgroup of $U^\infty(A)$.

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$CU^\infty(A)$ is the *closure* of the commutator subgroup of $U^\infty(A)$.

$\overline{K}_1^{\mathrm{alg}}(A)$ came up in Thomsen’s work on the role of the relationship between and K-theory and traces in classification theory.

Has seen lots of use in classification (e.g. in GLN).

Definition

$\underline{K}(A) = \bigoplus_{n=0}^{\infty} K_0(A; \mathbb{Z}/n\mathbb{Z}) \oplus K_1(A; \mathbb{Z}/n\mathbb{Z})$

Can think of $K_i(A; \mathbb{Z}/n\mathbb{Z})$ as $K_i(A \otimes \mathcal{O}_{n+1})$.

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Slogan

Can check "closeness" of $KK(\varphi)$ and $KK(\psi)$ by checking that $\underline{K}(\varphi)$ and $\underline{K}(\psi)$ agree on large finite subsets of $\underline{K}(A)$.

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A *compatible triple* $(\underline{\alpha}, \beta, \gamma)\colon\ \operatorname{inv}(A) \to \operatorname{inv}(E)$ consists of

$$\underline{\alpha}\colon \underline{K}(A) \to \underline{K}(E), \quad \beta\colon \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(E), \quad \gamma\colon\ \operatorname{Aff} T(A) \to \operatorname{Aff} T(E)$$

such that

$$\begin{tikzcd}
K_0(A) \arrow[r, "\rho_A"] \arrow[d, "\alpha_0"] & \operatorname{Aff} T(A) \arrow[r, "\mathrm{Th}_A"] \arrow[d, "\gamma"] & \overline{K}_1^{\mathrm{alg}}(A) \arrow[r] \arrow[d, "\beta"] & K_1(A) \arrow[d, "\alpha_1"] \\
K_0(B) \arrow[r, "\rho_B"] & \operatorname{Aff} T(B) \arrow[r, "\mathrm{Th}_B"] & \overline{K}_1^{\mathrm{alg}}(B) \arrow[r] & K_1(B)
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commutes.

$(\underline{\alpha}, \beta, \gamma)$ is *faithful and amenable* if $\gamma^*\big(T(B)\big) \subseteq T(A)$ consists of faithful amenable traces.

Theorem (C-Gabe-Schafhauser-Tikuisis-White)

- *A : sep., exact, UCT*
- *B : sep., $\mathcal{Z}$-stable, strict comparison w.r.t to $T(B)$, $T(B) \neq \varnothing$ & compact*
- *$(\underline{\alpha}, \beta, \gamma)\colon \mathrm{inv}(A) \to \mathrm{inv}(B)$: compatible triple that is faithful and amenable*

Then:

- *$\exists$ full† nuclear *-hom. $\varphi\colon A \to B$ s.t. $\mathrm{inv}(\varphi) = (\underline{\alpha}, \beta, \gamma)$;*
- *this φ is unique up to approx. unitary equivalence.*

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STRATEGY

Define $B_\infty := \ell^\infty B / c_0 B$.

Will prove classification of maps $A \to B_\infty$.

The *trace-kernel ideal* is

$$J_B := \Big\{ (x_n) \in B_\infty : \lim_{n\to\infty} \|x_n\|_{2,u} = 0 \Big\},$$

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Conditions on B $\rightsquigarrow$ J_B nice enough to employ KK machinery
$\rightsquigarrow$ B^∞ behaves like a finite vNa (sort of)

APPROXIMATE CLASSIFICATION OF MORPHISMS: MAJOR STEPS

$$A$$

$$0 \longrightarrow J_B \longrightarrow B_\infty \xrightarrow{q} B^\infty \longrightarrow 0$$

APPROXIMATE CLASSIFICATION OF MORPHISMS: MAJOR STEPS

$$\begin{tikzcd}
 & & A \arrow[dr, bend left, "1"] & & \\
0 \arrow[r] & J_B \arrow[r] & B_\infty \arrow[r, "q"] & B^\infty \arrow[r] & 0
\end{tikzcd}$$

1. Classify morphisms into B^∞
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APPROXIMATE CLASSIFICATION OF MORPHISMS: MAJOR STEPS

$$\begin{tikzcd} & & A \arrow[d, "2"] \arrow[dr, bend left, "1"] & & \\ 0 \arrow[r] & J_B \arrow[r] & B_\infty \arrow[r, "q"] & B^\infty \arrow[r] & 0 \end{tikzcd}$$

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2. Classify lifts of morphisms to B_∞
   (Ext, KK techniques)

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1. Classify morphisms into B^∞
   (von Neumann techniques)
2. Classify lifts of morphisms to B_∞
   (Ext, KK techniques)
3. Deal with KK-theory, exploiting J_B

Theorem (Castillejos-Evington-Tikuisis-White)

A: exact; B: $\mathbb{Z}$-stable, $T(B) \neq \varnothing$ & compact.
$\mathfrak{t}\colon T(B^\infty) \to T_{\mathrm{am}}(A)$ *continuous affine*
$\Longrightarrow \exists$ *nuclear* **-hom.* $\theta\colon A \to B^\infty$
s.t. $\tau\theta = \mathfrak{t}(\tau)\ \forall\, \tau \in T(B^\infty)$.
θ *is unique up to unitary equivalence.*

Compare with classifying maps from nuclear C^*-alg's into II_1 factors.

See Sam or Jorge's talk.

Setup: A: sep. exact; B: $\mathcal{Z}$-stable, strict comp. w.r.t. $T(B)$, $T(B) \neq \varnothing$ & compact

Theorem (Existence for lifts)

*$\theta\colon A \to B^\infty$ unitizably full nuclear *-hom;*
$\kappa \in KL_{\mathrm{nuc}}(A, B_\infty), \quad [q]_{KK_{\mathrm{nuc}}}\kappa = [\theta]_{KK_{\mathrm{nuc}}}$
$\implies \exists$ *unitizably full nuclear lift $\varphi\colon A \to B_\infty$ of θ s.t.* $[\varphi]_{KK_{\mathrm{nuc}}} = \kappa$.

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(Very) roughly:

- θ determines a pullback extension e_θ whose class in $\mathrm{Ext}_{\mathrm{nuc}}(A, J_B)$ vanishes.

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2. CLASSIFYING LIFTS (A GLIMPSE)

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- $[e_\theta] = 0 \implies e_\theta \oplus(\text{trivial extension}) \approx$ a split extension.
- Weyl-von Neumann-Voiculescu type absorption theorems $\implies e_\theta \oplus(\text{trivial extension}) \approx e_\theta$.

What if we have two lifts φ and ψ of θ?

$$\begin{tikzcd}
& & A \arrow[d, shift right, "\psi"'] \arrow[d, shift left, "\varphi"] \arrow[dr, bend left, "\theta"] & & \\
0 \arrow[r] & J_B \arrow[r] & B_\infty \arrow[r] & B^\infty \arrow[r] & 0
\end{tikzcd}$$

Want to guarantee (a strong form of) uniqueness with a condition that can be verified by comparing invariants.

Think of Voiculescu's Theorem:

$$\begin{tikzcd} & & A \arrow[d, shift right, "\psi"'] \arrow[d, shift left, "\varphi"] \arrow[dr, bend left, "\theta"] & & \\ 0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{B}(\mathcal{H}) \arrow[r] & \mathcal{Q}(\mathcal{H}) \arrow[r] & 0 \end{tikzcd}$$

If φ, ψ are "admissible" (faithful, nondegenerate, and $\varphi(A) \cap \mathcal{K} = \{0\} = \psi(A) \cap \mathcal{K}$), then $\varphi \approx_u \psi$.

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More can be said:

Theorem (Dadarlat-Eilers '01)

Suppose: $\varphi, \psi \colon A \to \mathcal{B}(\mathcal{H})$ *are admissible lifts of* θ.

Then:

$[\varphi, \psi] = 0 \in KK(A, \mathcal{K}) \implies \varphi \approx_u \psi$ *via unitaries in* $\mathcal{K} + \mathbb{C}1_{\mathcal{H}}$.

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Theorem (Uniqueness for lifts)

A: exact;

B: $\mathcal{Z}$-stable, strict comparison w.r.t. $T(B)$, $T(B) \neq \emptyset$ & compact;

φ, ψ: unitizably full, nuclear lifts of θ.

$$[\varphi, \psi] = 0 \in KL_{\mathrm{nuc}}(A, J_B) \quad \Longrightarrow \quad \varphi \approx_u \psi \text{ via unitaries in } \widetilde{J_B}\,.$$

See Jamie's talk.

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See Jamie's talk. (?)

3. From KL to $\overline{K}_1^{\mathrm{alg}}$: rotation maps

Solutions to lifting problems were encoded in certain KL_{nuc} classes. Want to compute this information in terms inv$(-)$.

Consider maps

$$\begin{tikzcd} & & A \arrow[d, shift right, "\psi"'] \arrow[d, shift left, "\varphi"] & & \\ 0 \arrow[r] & J_B \arrow[r] & B_\infty \arrow[r] & B^\infty \arrow[r] & 0 \end{tikzcd}$$

agreeing modulo J_B.

When does $[\varphi, \psi] \in KL_{\mathrm{nuc}}(A, J_B)$ vanish?

$\exists$ morphism

$$j_* \colon KL_{\mathrm{nuc}}(A, J_B) \to \mathrm{Hom}_\Lambda\left(\underline{K}(A), \underline{K}(B_\infty)\right)$$

$$[\varphi, \psi] \mapsto \underline{K}(\varphi) - \underline{K}(\psi)$$

induced by $j \colon J_B \to B_\infty$.

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Subtle obstruction: even if $\varphi(u) \sim_h \psi(u)$ is true for $u \in U(A)$, the path ξ connecting them might have nonzero "winding number".

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Subtle obstruction: even if $\varphi(u) \sim_h \psi(u)$ is true for $u \in U(A)$, the path ξ connecting them might have nonzero "winding number".

This leads to a rotation map $R\big([\varphi, \psi]\big)$ which (roughly) assigns the function

$$\tau \mapsto \frac{1}{2\pi i}\int_0^1 \tau\Big(\frac{d\xi(t)}{dt}\xi(t)^{-1}\Big)\, dt$$

on $T(B_\infty)$ to $[u] \in K_1(A)$. This is an element of $\mathrm{Aff}\, T(B_\infty)$.

Want to encode this additional obstruction to $[\varphi, \psi] = 0$ using $\overline{K}_1^{\mathrm{alg}}$.

Idea: use Thomsen's map

$$\operatorname{Aff} T(B_\infty) \to \overline{K}_1^{\mathrm{alg}}(B_\infty)$$

which, given $h = h^* \in B_\infty$, maps $\hat{h}$ to $[e^{2\pi i h}]_{\mathrm{alg}}$.

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Punchline

Assuming $\underline{K}(\varphi) = \underline{K}(\psi)$,

$$\overline{K}_1^{\mathrm{alg}}(\varphi) - \overline{K}_1^{\mathrm{alg}}(\psi) = 0 \quad \Longrightarrow \quad R\big([\varphi, \psi]\big) = 0 \quad \Longrightarrow \quad [\varphi, \psi] = 0.$$

This let us access the classification theorem for lifts.

SUMMARY

Using inv(—) can provide ∃ and ! for *-hom's

$$\boxed{\begin{array}{c}\text{sep., simple,}\\ \text{nuclear, UCT}\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{sep., simple, nuclear,}\\ \text{finite, } \mathcal{Z}\text{-stable}\end{array}}$$

(can do this much more generally!)

Using inv(—) can provide $\exists$ and ! for *-hom's

$$\boxed{\begin{array}{c}\text{sep., simple,}\\ \text{nuclear, UCT}\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{sep., simple, nuclear,}\\ \text{finite, } \mathcal{Z}\text{-stable}\end{array}}$$

(can do this much more generally!)

Can use this to classify algebras ... even in the non-unital setting.

Theorem

Suppose A and B are non-unital, simple, separable, nuclear, $\mathcal{Z}$-stable C^*-algebras satisfying the UCT.

Any isomorphism $\mathrm{Ell}^+(A) \xrightarrow{\simeq} \mathrm{Ell}^+(B)$ lifts to an isomorphism $A \xrightarrow{\simeq} B$.

Thank you!